1. In class, we learned that for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

the finite difference formulation is

$$u_{i+1,j} + u_{i-1,j} + r^2(u_{i,j+1} + u_{i,j-1}) - 2(1 + r^2)u_{i,j} = 0 \tag{1}$$

where $r = \Delta x / \Delta y$. The Jacobi Iteration Method can be used to solve the obove equation i.e.

$$u_{i,j}^{k+1} = \frac{1}{2(1 + r^2)} \left[u_{i+1,j}^k + u_{i-1,j}^k + r^2(u_{i,j+1}^k + u_{i,j-1}^k) \right] \tag{2}$$

where k corresponds to the previously computed values. The computation is carried out until a specified convergence criterion is achieved. The analogy between the iterative method and a time-dependent parabolic equation can be seen by considering the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

(a) Formulate the finite difference equation of the above PDE using FTCS formulation.

(b) Show that for $\Delta x = \Delta y$ it reduces to:

$$u_{i,j}^{n+1} = u_{i,j}^n + \frac{\Delta t}{(\Delta x)^2} \left[u_{i+1,j}^n + u_{i-1,j}^n - 4u_{i,j}^n + u_{i,j+1}^n + u_{i,j-1}^n \right] \tag{3}$$

(c) For stable solution

$$\frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{4}$$

By substituting this upper limit into above Eq. (3) show that the resulting equation is identical to the Jacobi iteration when r=1. State your conclusion.

2. Partial differential equations defined on circular boundaries can usually be solved more conveniently in polar coordinates than Cartesian coordinates, because they avoid the use of awkward difference analogs near the curved boundary. As an example of this, consider Laplace's equation in polar coordinates,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{4}$$

Now a grid in the $r\theta$ plane is defined by concentric circles $r = i(\Delta r)$ and the radial lines $\theta = j(\Delta\theta)$.

   (a) Discretize the above equation.

   (b) These equations can written in the following form

$$\mathbf{Au} = \mathbf{b}$$

   Is matrix $\mathbf{A}$ symmetric ?

3. Simplify Eq. (4) for a problem with symmetry with respect to origin.

4. The procedure described above causes no difficulty as long as $r \neq 0$. For $r = 0$ the second and third terms of Eq. (4) appear to become infinitely large. Therefore, if a solution to the problem at $r = 0$ is desired, the region around the point $r = 0$ should be given a special treatment. Show that the equation derived in problem (3) at $r = 0$ can be replaced by

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial r^2} = 2\frac{\partial^2 u}{\partial r^2} = 0 \qquad \text{at} \quad r = 0$$

5. How you resolve such a difficulty when the problem is not symmetric about the origin ?