1. Consider the following eigenvalue problem:

\[-\nu''(x) = \gamma \nu(x), \quad 0 < x < 1\]
\[\nu'(0) = 0 = \nu'(1)\]

(a) Show that employing one-sided finite difference scheme at the boundaries (backward and forward) as well as use of following discretization in the uniform grid \( h = 1/N \), \( i = 0, 1, 2, ..., N \), \( x_i = -h/2 + ih \) result in a matrix form equation depicted below:

\[-V_{n-1} + 2V_n - V_{n+1} = \lambda V_n, \quad n = 1, 2, ..., N\]
\[V_1 - V_0 = 0 = V_{N+1} - V_N\]

\[
\begin{bmatrix}
1 & -1 \\
-1 & 2 & -1 \\
& \ddots & \ddots & \ddots \\
-1 & 2 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_N
\end{bmatrix}
= \lambda
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_N
\end{bmatrix}
\]

(b) Prove that the eigenvalues of matrix A are as follow:

\[\lambda_n = 2 - 2 \cos \left( \frac{(n - 1) \pi}{N} \right), \quad n = 1, 2, ..., N\]

and the corresponding eigenvectors are:

(c) without any mathematical operations prove that \( V_n \cdot V_m = 0 \) for \( n \neq m \)

(hint: \( A = A^T \))

(d) show that \( V_1 \cdot V_1 = N, V_n \cdot V_n = N/2 \) for \( n = 2, 3, ..., N \)
\[ V_n = \begin{bmatrix} 
\cos(n - 1)\pi x_1 \\
\cos(n - 1)\pi x_2 \\
\vdots \\
\vdots \\
\cos(n - 1)\pi x_N 
\end{bmatrix} \quad n = 1, 2, \ldots, N \]