Convection, Diffusion and Dispersion Characteristics

Physical phenomena

Diffusion $f_t = \alpha f_{xx}$
Convection $f_t + uf_x = 0$
Dispersion $f_t = \beta f_{xxx}$

Convection

$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = 0$

u = Convection Velocity

Characteristics of a convection equation can be determined by investigating a simple wave moving in time and space.
Using complex Fourier, a simple wave is determined as below:

\[ F(x, t) = Ce^{i(\sigma + ikx + \omega t)} = \Re(F(x, t)) + i\Im(F(x, t)) \]

- \( C \) = Wave amplitude
- \( s = \sigma + i\omega \) = Complex wave frequency
- \( k \) = Wave Number
- \( \Re \) = Real Part
- \( \Im \) = Imaginary Part

Convection

Substituting in Convection Eq.

\[(\sigma + i\omega)Ce^{i(\sigma + ikx + \omega t)} + u(k)Ce^{i(\sigma + ikx + \omega t)} = 0 \]

\( \sigma + \omega(\omega + uk) = 0 \)
\( \sigma = \omega \)
\( \omega = -uk \)

\[ F(x, t) = Ce^{i(\omega t - \omega x)} \]

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Convection

\[ F(x,t) = Ce^{\text{ist}} = \Re(F(x,t)) + i\Im(F(x,t)) \]
\[ F(x,t) = Ce^{i(kx-\omega t)} \]

Consider a sine wave as initial condition

\[ f(x, t) = A_m \sin mx \]

\[ F(x, t) = Ce^{i(kx+mωt)} = C(\cos kx + i \sin kx) = A_m \sin mx \]
\[ C = A_m \]
\[ k = \frac{m\pi}{L} \]
\[ f(x, t) = A_m \sin(m(x-ut)) \]

Any wave is a linear superposition of Sine and Cosine waves with different amplitudes and wave lengths.

Above equations show that any Fourier component moves without any change in shape and with constant velocity.

Any arbitrary wave moves without any change in shape and with constant velocity.

Pure Convection equation only moves the initial distribution with constant velocity.
Diffusion

\[ f_t = \alpha f_{xx} \]
\[ \alpha = \text{Thermal diffusion coefficient} \]

\[ F(x,t) = Ce^{\sigma t} e^{i\omega t} = \Re(F(x,t)) + i\Im(F(x,t)) \]
\[ s = \sigma + i\omega \]

\[ F(x,t) = Ce^{\alpha t} e^{i(kx + \omega t)} \]

Substituting in Convection Eq. \( f = \alpha f_{xx} \)

\[ (\sigma + i\omega)Ce^{\alpha t} e^{i(kx + \omega t)} = \alpha(k)^2 Ce^{\alpha t} e^{i(kx + \omega t)} \]

Consider a sine wave as Initial Condition

\[ f(x,+) = A_m \sin m x \]

\[ F(x,t) = Ce^{\alpha t} \sin m x = e^{-\omega t/\lambda} (A_m \sin m x) = e^{-\omega t/\lambda} f(x,+) \]

\( \star \)
- The initial condition decreases exponentially \( e^{-\omega t/\lambda} \)
- The reduction rate of initial wave depends on \( \frac{\lambda}{\omega} \)
- The initial condition can not spread in the space

Every Fourier component decreases based on its wave number

The amplitude and shape of the initial wave changes with time
Diffusion

\[ \frac{1}{\alpha} = \gamma \pi \]

\[ k = \frac{\pi}{a} \]

\[ \gamma = \sqrt{\frac{\pi}{a}} \]

Substituting in Convection Eq.

\[ f_t = \beta f_{xxx} \quad \beta = \text{Dispersion coefficient} \]

\[ F(x, t) = C e^{\sigma t} e^{i(kx + \omega t)} \]

\[ s = \sigma + i\omega \]

\[ F(x, t) = C e^{\sigma t} e^{i(kx + \omega t)} \]

Dispersion

Substituting in Convection Eq.

\[ (\sigma + i\omega)Ce^{\sigma t} e^{i(kx + \omega t)} = \beta(ik)Ce^{\sigma t} e^{i(kx + \omega t)} \]

\[ \sigma + i\omega = -\beta k^2 \]

\[ \sigma = \sigma \]

\[ \omega = -\beta k^2 \]

\[ F(x, t) = C e^{\sigma t} e^{i(kx + \omega t)} \]
Consider a sine wave as Initial Condition

\[ f(x, t) = A_m \sin mx \]

\[ F(x, t) = C e^{ikx} e^{i\beta t} = \Re(F(x, t)) + i\Im(F(x, t)) \]

In pure convection equation Wave velocity is constant

\[ c = u \]

In dispersion equation Wave velocity is constant but depends on wave number

\[ c = \beta/k' \]

Each wave moves without any changes in its shape and amplitude but with different constant velocity

The dispersion equation spreads the initial wave in the space and changes its shape during solution time