Session 3

Contents

- **Well-posed and ill-posed problems**
- **Elliptic partial differential equations**
- **Parabolic partial differential equations**
- **Hyperbolic partial differential equations**

Well-posed and ill-posed problems:

Steady uniform temperature distribution in an infinite homogeneous isotropic plate with insulated surfaces:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (x, y) \in \Omega$$

$$u(x, y) = f(x, y), \qquad (x, y) \in \partial\Omega$$

In related references, it is shown that the above problem has unique solution, but to ensure finding a solution describing physical phenomena as well as possible, we must also consider the dependence between the solutions and problem data.

Suppose that u is solution of the problem with data f and u' is solution of the problem with the data f'.

The solution of problem depends continuously on the data problem, if

$$\mathrm{Max}_{(x,y)\in\Omega} |u(x, y) - u'(x, y)| < \epsilon$$

Provided that

$$\mathrm{Max}_{(x,y)\in\partial\Omega} |f(x, y) - f'(x, y)| < \delta$$

Well-posed and ill-posed problems:

A boundary value problem is well-posed under the conditions:

1) A solution exists
2) The solution is unique
3) The solution depends continuously on the boundary data

Definition (Hadamard 1952):

A boundary value problem is well-posed if and only if, it has a unique solution that depends continuously on the boundary data.

Hadamard's example:

Consider Laplace equation with mentioned condition

$$u_{xx} + u_{yy} = 0$$

$$u(x, 0) = 0, \qquad u_y(x, 0) = n^{-1} \sin nx$$

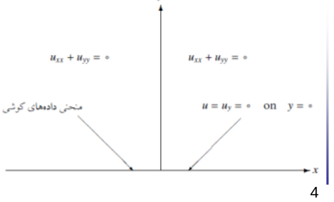
Solution: $u(x, y) = n^{-2} \sinh ny \sin nx$

As $n \to \infty$,

$u_y(x, 0) \to 0$ but

$u(x, y)$, $y \neq 0$ $\to \infty$

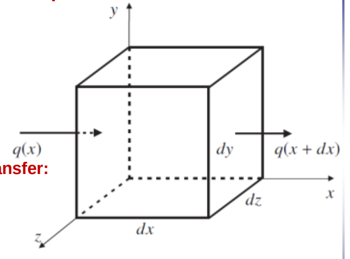
So the continuity with initial data is lost.



Elliptic partial differential equations:

2D Laplace equation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$



Steady conduction heat transfer:

$$q = -kA\frac{\partial T}{\partial n}$$

$$q\ \mathrm{Net}_{,x} = q(x) - q(x+dx)$$

$$= q(x) - \left(q(x) + \frac{\partial q(x)}{\partial x}dx\right) = -\frac{\partial q(x)}{\partial x}dx$$

5

$$\begin{cases} q\ \mathrm{Net}_x = -\dfrac{\partial}{\partial x}\left(-kA\dfrac{\partial T}{\partial x}\right)dx = \dfrac{\partial}{\partial x}\left(k\dfrac{\partial T}{\partial x}\right)d\forall \\ q\ \mathrm{Net}_y = \dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right)d\forall \\ q\ \mathrm{Net}_z = \dfrac{\partial}{\partial z}\left(k\dfrac{\partial T}{\partial z}\right)d\forall \end{cases}$$

According to energy conservation law:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) = 0$$

If $k \neq k(x, y, z, T) = \text{const}$

$$T_{xx} + T_{yy} + T_{zz} = \nabla^2 T = 0$$

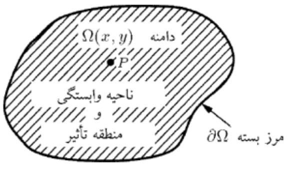
6

General properties of elliptic type problem:

$$T_{xx} + T_{yy} = 0$$

Boundary Conditions:

$$\begin{cases} T = \text{const} \\ T_n = \text{const} \\ \alpha T + \beta T_n = \text{const} \end{cases}$$



Will have the following properties:

1) No matter how bad is the B.C. are the solution is infinity smooth inside (even analytic).
2) Maximum principle and Minimum principle
3) Mean value property

7

Poisson's equation:

Steady conduction heat transfer with energy source:

$$\dot{E} = S(x, y, z)d\forall$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + S(x, y, z) = 0$$

If $k \neq k(x, y, z, T)$ $\longrightarrow$ $T_{xx} + T_{yy} + T_{zz} = \nabla^2 T = -\frac{S}{k}$

2D Steady conduction heat transfer:

$$T_{xx} + T_{yy} = 0$$

$A = 1,\ B = 0$ and $C = 1$ $\xrightarrow{\text{Equation classification}}$ $B^2 - 4AC = -4 < 0$

slope of characteristic curve: $\frac{dy}{dx} = \pm\sqrt{-1}$

8

Parabolic partial differential equations:

Diffusion equation:

$$f_t = \alpha \nabla^2 f$$

Unsteady conduction heat transfer:

$$dE_{\text{stored}} = dmCT = \rho d\forall CT = \rho CT\, d\forall$$

$$\frac{\partial(\rho CT)}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

If $k \neq k(x, y, z, T) = \text{const}$ →

$$T_t = \alpha(T_{xx} + T_{yy} + T_{zz}) = \alpha \nabla^2 T \qquad \alpha = (k/\rho C)$$

1D Unsteady conduction heat transfer:

$$T_t = \alpha T_{xx}$$

$A = \alpha,\ B = 0$ and $C = 0$ —Equation classification→ $B^2 - 4AC = 0$

characteristics:

$dt = 0$ → $t =$ ثابت

Data transmission speed along the characteristics

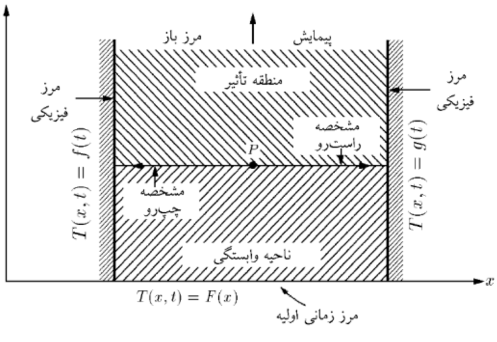
$$c = \frac{dx}{dt} = \frac{dx}{\pm 0} = \pm\infty$$

General properties of parabolic type problem:

We expect:

1) Solution is infinitely differentiable in $x, y, \dots, t$ for $t > 0$ (inside)
2) Infinite signal speed
3) Maximum of solution $u(x, y, t)$ in $0 \le t \le T$ is at $t = 0$ or at $t = T$
4) Normally expect “monotone convergence” to steady state (when steady state exists) i.e. as $t \to \infty$
5) Events propagate into the future

Solution domain for diffusion parabolic partial differential equations:



1D Unsteady diffusion equations:

$u_t = \beta u_{xx}$

Boundary Conditions:

$$\begin{cases} u(x_0, t) = f(t) \\ u(x_{end}, t) = g(t) \end{cases} \quad \text{or} \quad \begin{cases} u_x(x_0, t) = f(t) \\ u_x(x_{end}, t) = g(t) \end{cases}$$

Initial guess: $u(x, t_0) = F(x)$

Steady diffusion problems:

$u_y = \beta u_{xx}$

Initial guess: $u(x, y_0) = F(x)$

$\begin{cases} y\text{: Time-like direction} \\ x\text{: Space-like direction} \end{cases}$

Hyperbolic partial differential equations:

Wave equation:

$f_{tt} = c^2 \nabla^2 f$

Wave equation for linearized acoustic domain:

$u'_{tt} = a_0^2 u'_{xx}$ $\qquad$ $p'_{tt} = a_0^2 p'_{xx}$

$A = 1,\ B = 0 \text{ and } C = -a_0^2$ $\xrightarrow{\text{Equation classification}}$ $B^2 - 4AC = 4a_0^2 > 0$

Data transmission speed along the characteristics

$c = \dfrac{dx}{dt} = \pm a_0$ $\longrightarrow$ $x = x_0 \pm a_0 t$

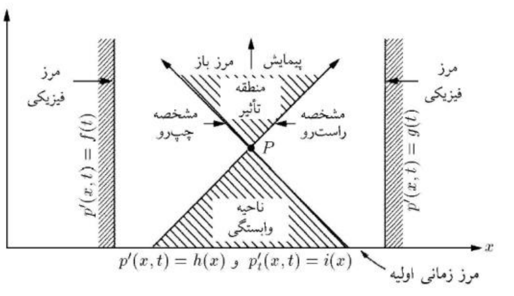
General properties of hyperbolic type problem:

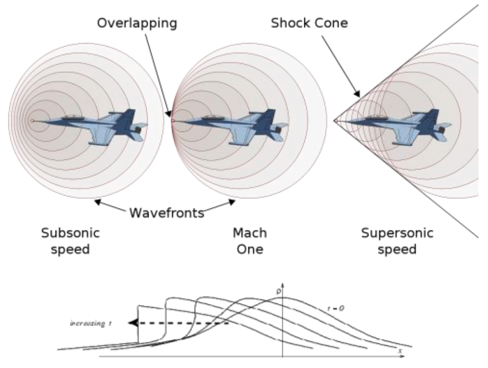
Will have the following properties:

1) Values of the solution depend locally on the initial data
2) finite signal speed
3) Discontinuities can propagate
4) Continues boundary and initial values can give rise to discontinuity solution.
5) Solution is no more continues than the initial or boundary condition

(3), (4) & (5) for shock wave

Solution domain for wave hyperbolic partial differential equations:





17

Two simultaneous equations of displacement first order

$$\begin{cases} f_t + a g_x = 0 \\ g_t + a f_x = 0 \end{cases} \longrightarrow f_{tt} = a^2 f_{xx}$$

Boundary Conditions:

$$\begin{cases} u(x_0, t) = f(t) \\ u(x_{end}, t) = g(t) \end{cases} \quad \text{or} \quad \begin{cases} u_x(x_0, t) = f(t) \\ u_x(x_{end}, t) = g(t) \end{cases}$$

Initial guess:

$$\longrightarrow \begin{cases} f(x, t_0) = F(t) \\ f_t(x, t_0) = F(t) \end{cases}$$

18

$$\begin{cases} \rho u_t + \rho u u_x + p_x = 0 \\ \rho a^2 u_x + p_t + u p_x = 0 \\ du = u_t dt + u_x dx \\ dp = p_t dt + p_x dx \end{cases} \longrightarrow \begin{bmatrix} \rho & \rho u & 0 & 1 \\ 0 & \rho a^2 & 1 & u \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{bmatrix} u_t \\ u_x \\ p_t \\ p_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ du \\ dp \end{bmatrix}$$

$$-\rho(dx)(dx - udt) + (dt)\left(\rho u(dx - udt) + \rho a^2 dt\right) = 0$$

$$(dx)^2 - 2u(dx)(dt) + (u^2 - a^2)(dt)^2 = 0$$

Data transmission speed along the characteristics

$$c = \frac{dx}{dt} = u \pm a$$

19